Math 429 - Exercise Sheet 6

1. Show that a one-dimensional representation of any Lie algebra $\mathfrak{g}$ is the same as a covector of

$$\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$$

(i.e. a linear map from the above vector space to the ground field).

2. Use the Poincaré-Birkhoff-Witt theorem to show that the universal enveloping algebra $U\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$ (over any field) has no zero-divisors.

3. Suppose we have a Lie algebra $\mathfrak{g}$ over a field of characteristic zero. While the assignment

$$S\mathfrak{g} \to U\mathfrak{g}, \qquad x_1 \dots x_n \to \frac{1}{n!} \sum_{\sigma \in S(n)} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$$

is not an algebra homomorphism, show that it is a isomorphism of (infinite-dimensional) representations of $\mathfrak{g}$. *First you'll have to make $S\mathfrak{g}$ and $U\mathfrak{g}$ into representations of $\mathfrak{g}$: use the adjoint representation and equation (31) in the lecture notes.*

4. Show that given abelian Lie algebras $\mathfrak{h}$ and $\mathfrak{h}'$, there is a one-to-one correspondence between Lie algebras $\mathfrak{g}$ such that

$$\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h} \qquad \text{and} \qquad \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h}'$$

(such $\mathfrak{g}$ are called **2-step nilpotent**) and non-degenerate anti-symmetric bilinear forms

$$\mathfrak{h}' \times \mathfrak{h}' \to \mathfrak{h}$$

5. Consider any matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n$ (for some n large enough). Show that

$$\mathfrak{h} = \mathfrak{g} \cap \Big\{\text{diagonal } n \times n \text{ matrices}\Big\}$$
$$\mathfrak{n} = \mathfrak{g} \cap \Big\{\text{strictly upper triangular } n \times n \text{ matrices}\Big\}$$
$$\mathfrak{b} = \mathfrak{g} \cap \Big\{\text{upper triangular } n \times n \text{ matrices}\Big\}$$

are abelian, nilpotent, solvable Lie subalgebras of $\mathfrak{g}$ (respectively).